Journal of Engineering Physics and Thermophysics, Vol. 75, No. 6, 2002

ANALYTICAL CALCULATION OF SOLIDIFICATION OF A MELT IN A MOLD

V. V. Dremov and F. V. Nedopekin

UDC 539.19

The mathematical method of investigation of the process of crystallization of a molten metal in a two-dimensional mold has been developed using the variational formulation of the problem with determination of the velocity of motion of the crystallization front and the solidifed-metal thickness at any instant of time.

In connection with the progress made in computational engineering, in considering complicated heat- and mass-transfer problems one gives preference to numerical methods of solution which make it possible to take into account many factors and to obtain an adequate description of specific considered phenomena [1–3]. But apart from advantages, these methods have considerable disadvantages, among which are the absence of clarity in obtaining a solution and the inability to represent it in the form of an analytical function describing the behavior of a given system. The existing analytical solutions of crystallization problems are mainly considered in a one-dimensional formulation, with rare exception [4, 5] when simple two-dimensional problems are examined. Therefore, attempts to solve the two-dimensional problems of heat transfer by analytical methods remain topical.

In the present work, it is proposed to solve the problem of crystallization of a metal in a mold by the variational method of local potential [6] with the subsequent application of the method of partial integration. The combined application of these methods allowed us to derive a rather simple approximate solution for the velocity of motion of the crystallization front in the form of explicit functions of the coordinates and time.

Let progressive solidification (crystallization) be carried out in a wedge-shaped region which is formed by two planes located at a small angle 2α. At the top and at the bottom, the region is bounded by cylindrical surfaces of radii R_1 and R_2. To solve the problem, we select a cylindrical coordinate system (r, φ, z). On the side surface $\varphi = \alpha$ and on the cylindrical surface $r = R_1$, the temperature $T = T_{\text{sur}} = \text{const}$ is assumed to be lower than the crystallization temperature T_{cr}. On the surface $r = R_2$, we take $T_{\text{in}} = \text{const}$. The plane $\varphi = \alpha$ bounds the crystallization region on the side, and for $t \geq 0$ it has the temperature T_{sur}. When $t > 0$, the process of crystallization begins and we have $T = T_{\text{cr}}$ at the crystallization front.

The problem is considered to be infinite in z; therefore, the temperature and the velocity are independent of z. Since the angle between the axis and the side mold wall is small, we neglect the transverse velocity component V_φ. Then the heat-transfer equation [7] written for the region of the molten metal takes the form

$$\rho_1 c_{v1}\left(\frac{\partial T_1}{\partial t} + V_r \frac{\partial T_1}{\partial r}\right) = \lambda_1 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T_1}{\partial \varphi^2}\right]. \tag{1}$$

Here $0 < \varphi < \varphi_{\text{fr}}$ and $r_{\text{fr}} < r < R_2$.

A similar equation at $V_r = 0$ will describe the temperature field in the solid phase

$$\rho_2 c_{v2}\left(\frac{\partial T_2}{\partial t}\right) = \lambda_2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T_2}{\partial \varphi^2}\right] \tag{2}$$

for the region $\varphi_{\text{fr}} < \varphi < \alpha$ and $R_1 < r < r_{\text{fr}}$.

Donbas State Academy of Civil Engineering and Architecture, Makeevka, Ukraine; email: dremov@tr.dn.ua. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 75, No. 6, pp. 179–184, November–December, 2002. Original article submitted January 17, 2002; revision submitted May 15, 2002.

The thickness of the solidified skin for any radius can be found from the formula

$$\varepsilon (r_{fr}, \varphi_{fr}, t_{fr}) = r_{fr} (t_{fr}) [\alpha - \varphi_{fr} (t_{fr})] . \tag{3}$$

It is considered that the chord coincides with the arc at small arc lengths, which corresponds to a small angle α.

At the instant $t = 0$, the solid phase is absent, and $T_1(r, \varphi, 0) = T_{in}$ for $R_2 > r > R_1$ and $\alpha > \varphi > 0$. When $\varphi = \alpha$, $r = R_1$, and $t > 0$, we have

$$T_1 (r_{fr}, \varphi_{fr}, t_{fr}) = T_{cr} , \tag{4}$$

$$T_2 (r, \alpha, t) = T_{sur} . \tag{5}$$

The thermal contact at the crystallization front is assumed to be ideal; therefore, for $r = r_{fr}(t)$ and $\varphi = \varphi_{fr}(t)$ the temperatures of the liquid and the solid phase are equal to the crystallization temperature:

$$T_1 (r_{fr}, \varphi_{fr}, t_{fr}) = T_2 (r_{fr}, \varphi_{fr}, t_{fr}) = T_{cr} . \tag{6}$$

The latent crystallization heat L_1 is released at the moving front of phase transition; this heat is removed through the solid phase together with superheating [4]. Consequently,

$$\lambda_2 \left(\frac{1}{r} \frac{\partial T_2}{\partial \varphi} \right)_{fr} = L_1 \rho \frac{\partial \varepsilon}{\partial t} + \lambda_1 \left(\frac{1}{r} \frac{\partial T_1}{\partial \varphi} \right)_{fr} . \tag{7}$$

This equation of heat balance at the crystallization front is used to determine $\varepsilon(t)$.

Three unknown functions, $T_1(r, \varphi, t)$, $T_2(r, \varphi, t)$, and $\varepsilon(t)$, should be found from Eqs. (1)–(3) and boundary conditions (4)–(7).

Equation (1) can be rewritten as follows:

$$\frac{\partial T_1}{\partial t} + V_r \frac{\partial T_1}{\partial r} = a_1 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_1}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T_1}{\partial \varphi^2} \right] . \tag{8}$$

The exact solution of Eq. (8) in r has the form

$$T_1 = \frac{(T_{in} - T_{cr}) \ln r + T_{cr} \ln R_2 - T_{in} \ln r_{fr}}{\ln (R_2 / r_{fr})} . \tag{9}$$

To simplify the complete solution of Eq. (8) we approximate the exact solution by a rough one in the form of a linear function in r which satisfies the boundary conditions

$$T_1 = T_{cr} \ \text{ for } \ r = r_{fr} , \quad T_1 = T_{in} \ \text{ for } \ r = R_2 . \tag{10}$$

Then we obtain

$$T_1 = T_{in} - \frac{R_2 - r}{R_2 - r_{fr}} (T_{in} - T_{cr}) . \tag{11}$$

From the comparison of the temperatures obtained from Eq. (11) and the temperatures obtained from Eq. (9) it is evident that the former are lower by 0.5% than the exact values in the middle of the interval (R_1, R_2) and that they coincide with the latter on the boundaries.

We seek the approximate solution of (8) by using the variational method [6] and gradually complicate the problem. At first, we find the dependence for φ for the stationary case:

$$V_r \frac{\partial T_1}{\partial r} = a_1 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_1}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T_1}{\partial \varphi^2} \right] . \tag{12}$$

Omitting the subscript 1 on the temperature and denoting the derivatives in the form of $T_r = \partial T/\partial r$, $T_{\varphi\varphi} = \partial^2 T/\partial \varphi^2$, and $T_{rr} = \partial^2 T/\partial r^2$, we obtain

$$\frac{V_r}{a_1} r T_r - T_r - r T_{rr} - \frac{1}{r} T_{\varphi\varphi} = 0 . \tag{13}$$

The functional corresponding to Eq. (13) will be written in the form

$$\int_{r_{\text{fr}}}^{R_2} \int_0^{\varphi_{\text{fr}}} \left[2 \frac{V_r}{a_1} r T_r^0 T + r T_r^2 + \frac{1}{r} T_\varphi^2 \right] dr d\varphi = L , \tag{14}$$

where $T_r^0 = \partial T^0/\partial r$ and the superscript 0 on T_r denotes the unvaried derivative of the temperature.

Now we verify the fact that the variation of L with respect to T does give Eq. (13). To do this, we write the Euler–Lagrange equation corresponding to Eq. (14)

$$\frac{\partial L}{\partial T} - \frac{\partial}{\partial r} \frac{\partial L}{\partial T_r} - \frac{\partial}{\partial \varphi} \frac{\partial L}{\partial T_\varphi} = 0 , \tag{15}$$

and calculate the corresponding derivatives

$$\frac{\partial L}{\partial T} = 2 \frac{V_r}{a_1} r T_r^0 , \quad \frac{\partial L}{\partial T_r} = 2 r T_r , \quad \frac{\partial}{\partial r} \frac{\partial L}{\partial T_r} = 2 T_r + 2 r T_{rr} ,$$

$$\frac{\partial L}{\partial T_\varphi} = \frac{2}{r} T_\varphi , \quad \frac{\partial}{\partial \varphi} \frac{\partial L}{\partial T_\varphi} = \frac{2}{r} T_{\varphi\varphi} . \tag{16}$$

Substituting Eq. (16) into Eq. (15), reducing it by two, and omitting the superscript 0 on T_r, we obtain Eq. (13). Consequently, indeed, functional (14) corresponds to Eq. (13) and the function minimizing it will be the best approximation of the solution of (13). Then we seek the function minimizing functional (14) in the form

$$T = T(r) f(\varphi) = \left[T_{\text{in}} - \frac{R_2 - r}{R_2 - r_{\text{fr}}} (T_{\text{in}} - T_{\text{cr}}) \right] f(\varphi) . \tag{17}$$

The calculation of the derivatives with respect to r and φ from Eq. (17) gives

$$T_r = \frac{T_{\text{in}} - T_{\text{cr}}}{R_2 - r_{\text{fr}}} f(\varphi) , \quad T_\varphi = \left[T_{\text{in}} - \frac{R_2 - r}{R_2 - r_{\text{fr}}} (T_{\text{in}} - T_{\text{cr}}) \right] f'(\varphi) . \tag{18}$$

Substituting Eqs. (17) and (18) into Eq. (14) and integrating over r, we find

$$\int_0^{\varphi_{\text{fr}}} \left\{ A_1 f^0(\varphi) f(\varphi) + E f^2(\varphi) + B [f'(\varphi)]^2 \right\} d\varphi = L . \tag{19}$$

The function $f(\varphi)$ is selected so that integral (19) is minimum. This condition will be satisfied by the function making the variation

$$\delta L = \frac{\partial L}{\partial f(\varphi)} - \frac{\partial}{\partial \varphi} \frac{\partial L}{\partial f'(\varphi)} = 0 \tag{20}$$

vanish. Having taken the variation, we obtain

$$f''(\varphi) - \frac{A}{B} f(\varphi) = 0\ . \tag{21}$$

The following function [8] will be the solution of Eq. (21):

$$f(\varphi) = C_1 \cosh \varphi \sqrt{\frac{A}{B}} + C_2 \sinh \varphi \sqrt{\frac{A}{B}}\ . \tag{22}$$

The constants C_1 and C_2 can be determined from the boundary conditions for the function $f(\varphi)$: $T = T_{\text{cr}}$ for $\varphi = \varphi_{\text{fr}}$ and $\partial T/\partial \varphi = 0$ for $\varphi = 0$. Taking into account these constants in Eq. (22) and calculating, we obtain $C_2 = 0$ and $C_1 = 1\ (\cosh \varphi_{\text{fr}} \sqrt{A/B})$. Substituting them into Eq. (22) and the latter into Eq. (17), we find the solution of Eq. (12):

$$T = \frac{[T_{\text{in}}(r - r_{\text{fr}}) + T_{\text{cr}}(R_2 - r)]}{R_2 - r_{\text{fr}}} \frac{\cosh \varphi \sqrt{\frac{A}{B}}}{\cosh \varphi_{\text{fr}} \sqrt{\frac{A}{B}}}\ . \tag{23}$$

A search for the solution of a complete nonstationary heat-conduction equation is performed by a method identical to that in determining the dependence for φ. The functional corresponding to Eq. (8) has the form

$$L = \int_0^{t_{\text{fr}}} \int_0^{\varphi_{\text{fr}}} \int_{r_{\text{fr}}}^{R_2} \left[\frac{2V_r}{a_1} r T_r^0 T + 2 \frac{r}{a_1} T T_t^0 + r T_r^2 + \frac{1}{r} T_\varphi^2 \right] dr d\varphi dt\ . \tag{24}$$

The solution of (8) is sought in the form of the function (23) multiplied by the unknown function $f(t)$:

$$T = \frac{[T_{\text{in}}(r - r_{\text{fr}}) + T_{\text{cr}}(R_2 - r)]}{R_2 - r_{\text{fr}}} \frac{\cosh \varphi \sqrt{\frac{A}{B}}}{\cosh \varphi_{\text{fr}} \sqrt{\frac{A}{B}}} f(t)\ . \tag{25}$$

Substitution of Eq. (25) into Eq. (24) and integration over r and φ gives

$$L = \int_0^{t_{\text{fr}}} \left\{ K f^0(t) f(t) + M f(t) [f'(t)]^0 + P f^2(t) + Q f^2(t) \right\} dt\ . \tag{26}$$

Having taken the variation of Eq. (26) with respect to $f(t)$ and having equated it to zero, we obtain

$$M f'(t) + G_1 f(t) = 0\ .$$

The solution of this equation will be the function

$$f(t) = C \exp(-G_1 t/M) . \tag{27}$$

The constant C can be found from the condition at the crystallization front

$$T_1 (r = r_{\text{fr}}, \varphi = \varphi_{\text{fr}}, t = t_{\text{fr}}) = T_{\text{cr}} . \tag{28}$$

For this, we write the solution of the nonstationary problem

$$T_1 = \frac{[T_{\text{in}} (r - r_{\text{fr}}) + T_{\text{cr}} (R_2 - r)]}{R_2 - r_{\text{fr}}} \frac{\cosh \varphi \sqrt{\dfrac{A}{B}}}{\cosh \varphi_{\text{fr}} \sqrt{\dfrac{A}{B}}} C \exp\left(-\frac{G_1}{M} t\right). \tag{29}$$

Taking into account Eq. (28) in Eq. (29), we obtain $C = \exp(G_1 t_{\text{fr}}/M)$. Consequently, the solution of Eq. (8) will be the function

$$T_1 = \frac{[T_{\text{in}} (r - r_{\text{fr}}) + T_{\text{cr}} (R_2 - r)]}{R_2 - r_{\text{fr}}} \frac{\cosh \varphi \sqrt{\dfrac{A}{B}}}{\cosh \varphi_{\text{fr}} \sqrt{\dfrac{A}{B}}} \exp\left[-\frac{G_1}{M} (t - t_{\text{fr}})\right]. \tag{30}$$

In just the same way, we find the temperature distribution in the hard region of an ingot. To do this, we divide Eq. (2) by $\rho_2 c_{v2}$ and introduce the thermal diffusivity of the hard metal: $a_2 = \lambda_2/(\rho_2 c_{v2})$. With allowance for the above, Eq. (2) will take the form

$$\frac{\partial T_2}{\partial t} = a_2 \left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T_2}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 T_2}{\partial \varphi^2}\right]. \tag{31}$$

Applying the boundary and initial conditions in the form of $T_2 = T_{\text{cr}}$ for $r = r_{\text{fr}}$, $\varphi = \varphi_{\text{fr}}$, and $t = t_{\text{fr}}$ and $T_2 = T_{\text{sur}}$ for $\varphi = \alpha$ and $r = R_1$ and using the variational formulation in combination with the method of partial integration, we obtain

$$T_2 = \left[T_{\text{cr}} - \frac{r_{\text{fr}} - r}{r_{\text{fr}} - R_1} (T_{\text{cr}} - T_{\text{sur}})\right] \frac{\sinh\left[(\alpha - \varphi) \sqrt{\dfrac{A_2}{B_2}}\right] + \sinh\left[(\varphi - \varphi_{\text{fr}}) \sqrt{\dfrac{A_2}{B_2}}\right]}{\sinh\left[(\alpha - \varphi_{\text{fr}}) \sqrt{\dfrac{A_2}{B_2}}\right]} \exp\left[-\frac{2G_2 (t - t_{\text{fr}})}{F_2}\right]. \tag{32}$$

To calculate the crystallization rate we rewrite Eq. (7) in the form

$$\frac{d\varepsilon}{dt} = \frac{1}{L_1 \rho}\left[\lambda_2 \left(\frac{1}{r} \frac{\partial T_2}{\partial \varphi}\right)_{\text{fr}} - \lambda_1 \left(\frac{1}{r} \frac{\partial T_1}{\partial \varphi}\right)_{\text{fr}}\right]. \tag{33}$$

TABLE 1. Coordinates of the Crystallization Front at Different Instants of Time

φ_{fr}	0°	0.25°	1°	2°	3°	4°	5°	6°	7°	9°	9.75°
m, m/sec$^{0.5}$	$2.82 \cdot 10^{-3}$	$3.35 \cdot 10^{-2}$	$6.7 \cdot 10^{-2}$	$9 \cdot 10^{-2}$	0.1	0.11	0.13	0.14	0.16	0.29	0.57
m^2	$8 \cdot 10^{-6}$	$11 \cdot 10^{-4}$	$4.4 \cdot 10^{-3}$	$8 \cdot 10^{-3}$	$1.1 \cdot 10^{-2}$	$1.3 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	$2.7 \cdot 10^{-2}$	$8 \cdot 10^{-2}$	0.33
r_{fr}, $t_{\text{fr}} = 0$	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
10 sec	1.2	1.205	1.218	1.233	1.244	1.254	1.265	1.28	1.31	1.5	2.17
50 sec	1.2002	1.223	1.29	1.36	1.41	1.45	1.5	1.57	1.67	2.35	
100 sec	1.2003	1.25	1.37	1.5	1.58	1.66	1.75	1.86	2.04		
200 sec	1.2007	1.29	1.52	1.75	1.89	2.02	2.16	2.34	2.62		
300 sec	1.201	1.33	1.66	1.96	2.16	2.32	2.5				
400 sec	1.2013	1.37	1.79	2.16	2.39						
600 sec	1.202	1.45	2.03	2.34							
3000 sec	1.21	2.19									

Then we calculate the derivatives with respect to φ from expressions (30) and (32) and substitute them into Eq. (33). The application of the condition at the crystallization front (3) allows us to determine the crystallization rate on the radius:

$$r_{\text{fr}} \frac{dr_{\text{fr}}}{dt} (\alpha - \varphi_{\text{fr}}) = \frac{T_{\text{cr}}}{L_1 \rho} \left\{ -\lambda_2 \sqrt{\frac{A_2}{B_2}} \frac{1 - \cosh \left[(\alpha - \varphi_{\text{fr}}) \sqrt{\frac{A_2}{B_2}} \right]}{\sinh \left[(\alpha - \varphi_{\text{fr}}) \sqrt{\frac{A_2}{B_2}} \right]} + \lambda_1 \sqrt{\frac{A}{B}} \frac{\sinh \varphi_{\text{fr}} \sqrt{\frac{A}{B}}}{\cosh \varphi_{\text{fr}} \sqrt{\frac{A}{B}}} \right\} . \tag{34}$$

Now we give the form of the expressions A, B, A_2, and B_2 required for numerical calculations:

$$A = \frac{V_r}{2a_1} (T_{\text{in}} - T_{\text{cr}}) \left[T_{\text{in}} (R_2 + r_{\text{fr}}) - \frac{T_{\text{in}} - T_{\text{cr}}}{3} (R_2 + 2r_{\text{fr}}) \right] + \frac{(T_{\text{in}} - T_{\text{cr}})^2}{2} \frac{R_2 + r_{\text{fr}}}{R_2 - r_{\text{fr}}} ,$$

$$B = T_{\text{in}}^2 \ln \frac{R_2}{r_{\text{fr}}} - 2T_{\text{in}} (T_{\text{in}} - T_{\text{cr}}) \left(\frac{R_2}{R_2 - r_{\text{fr}}} \ln \frac{R_2}{r_{\text{fr}}} - 1 \right) + (T_{\text{in}} - T_{\text{cr}})^2 \left[\frac{R_2^2 \ln \frac{R_2}{r_{\text{fr}}}}{(R_2 - r_{\text{fr}})^2} - \frac{3R_2 - r_{\text{fr}}}{2 (R_2 - r_{\text{fr}})} \right] ,$$

$$A_2 = \frac{(r_{\text{fr}} + R_1) (T_{\text{cr}} - T_{\text{sur}})^2}{2 (r_{\text{fr}} - R_1)} ,$$

$$B_2 = T_{\text{cr}}^2 \ln \frac{r_{\text{fr}}}{R_1} - 2T_{\text{cr}} (T_{\text{cr}} - T_{\text{sur}}) \left(\frac{r_{\text{fr}}}{r_{\text{fr}} - R_1} \ln \frac{r_{\text{fr}}}{R_1} - 1 \right) + (T_{\text{cr}} - T_{\text{sur}})^2 \left[\frac{r_{\text{fr}}^2}{(r_{\text{fr}} - R_1)^2} \ln \frac{r_{\text{fr}}}{R_1} - \frac{3r_{\text{fr}} - R_1}{2 (r_{\text{fr}} - R_1)} \right] .$$

By fixing $(\alpha - \varphi_{\text{fr}})$ in Eq. (34), we obtain the dependence $r_{\text{fr}}(t)$ at the given value of the angle:

$$r_{\text{fr}} \frac{dr_{\text{fr}}}{dt} = \frac{T_{\text{cr}}}{L_1 \rho (\alpha - \varphi_{\text{fr}})} C_* , \tag{35}$$

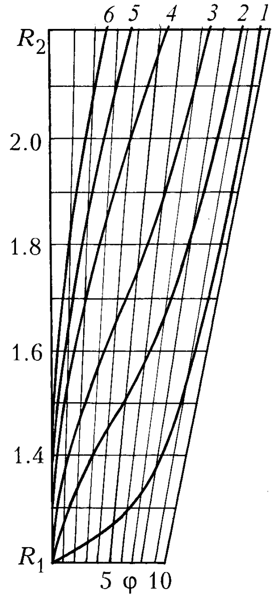

Fig. 1. Position of the crystallization front in the right half of the mold at different instants of time: 1) 10, 2) 50, 3) 100, 4) 200, 5) 300, and 6) 400 sec after the beginning of crystallization.

where C_* denotes the expression in the braces of Eq. (34).

Solution of Eq. (35) yields

$$r_{\text{fr}} = \sqrt{\frac{2T_{\text{cr}}C_*}{L_1\,\rho\,(\alpha - \varphi_{\text{fr}})}\,t + R_1^2}\ , \tag{36}$$

where it is taken into account that R_1^2 = const for $r_{\text{fr}} = R_1$ and $t = 0$.

For different angles φ_{fr} and time instants we calculated the coordinates of the crystallization front of the steel ingot for the following numerical values of the dimensions of the mold and the parameters of the molten and hard steel: R_1 = 1.2 m, R_2 = 2.2 m, $\alpha = 10^{\text{o}}$, T_{in} = 1833 K, T_{cr} = 1733 K, T_{sur} = 1433 K, $V_r = 10^{-2}$ m/sec, $a_1 = 4.5{\cdot}10^{-6}$ m^2/sec, λ_1 = 26.5 W/(m·K), λ_2 = 30.3 W/(m·K); $\rho = 7.31{\cdot}10^3$ kg/m^3, and $L_1 = 2.72{\cdot}10^5$ J/kg. The results of the calculation of $r_{\text{fr}}(t)$ are presented in Table 1; when $t_{\text{fr}} = 0$, r_{fr} = 1.2 m for all the values of φ_{fr}.

Analyzing the calculation results, we can draw the following conclusions: a) according to our calculations, the solidification constant $m = 2.82{\cdot}10^{-3}$ m/sec$^{0.5}$ at $\varphi_{\text{fr}} = 0$ coincides well with that obtained experimentally by Nelson [3] using the method of pouring out a liquid residue, and it is $m = 2.9{\cdot}10^{-3}$ m/sec$^{0.5}$ at the beginning of the process; b) the values presented in Table 1 for other angles are not the solidification constants, since these values characterize the position of the coordinate r_{fr} at the given value of the angle φ_{fr} at the instant of time t_{fr}. It is evident from Table 1 that the total time of solidification of this ingot is equal to 50 min.

From the results of the calculations we plotted the graphs of the position of the crystallization front at the corresponding instant of time. From the graphs it is seen that the bulk of the metal is crystallized in the first 5 min, whereas a narrow and deep liquid hole is crystallized rather slowly during the subsequent 45 min. The behavior of the crystallization front that moves more rapidly in the direction from the angle of the mold is also noteworthy. The narrow and deep liquid hole in the middle of the ingot, formed at the end of crystallization, can lead to the formation of shrinkage cavities on the ingot axis.

The dimensions of the trapezoidal ingot considered in the numerical example are as follows: $l_2 = R_2 \sin 2\alpha = 2.2 \sin 20^{\text{o}} = 0.752$ m and $l_1 = R_1 \sin 2\alpha = 1.2 \sin 20^{\text{o}} = 0.41$ m, height $R_2 - R_1 = 2.2 - 1.2 = 1$ m, length l = 1 m, mass 4.25 tons, and crystallization time 50 min. The figures of the position of the crystallization front of large slab ingots of height 2.5 m, width 0.75 m, length 1.7 m, and mass 23 tons are presented in [9]. It is shown that about half the ingot is solidified in 54 min, which almost coincides with our calculations of the total crystallization time of an ingot 2 times smaller in height, 1.7 times in length, and 5 times in mass.

The authors express their gratitude to S. P. Bakumenko for a discussion of the results of the present work.

NOTATION

α, cone angle of the side mold walls, deg; l_1 and l_2, width of the ingot at the bottom and at the top; R_1 and R_2, lower and upper radii of the mold, m; r, φ, z, cylindrical coordinates of the points inside the mold; T_1 and T_2, temperatures inside the molten and hard metals, K; T_{cr}, crystallization temperature, K; T_{sur}, temperature of the bottom and the side surface of the mold, K; T_{in}, initial temperature of pouring, K; t, time, sec; V_φ and V_r, azimuth and radial velocity components of the molten metal, m/sec, ρ_1 and ρ_2, densities of the molten and hard metals, kg/m^3; ρ, mean density of the metal, kg/m^3; c_{v1} and c_{v2}, specific heats of the molten and hard metals, J/(kg·K); λ_1 and λ_2, thermal conductivities of the molten and hard metals, W/(m·K); φ_{fr} and r_{fr}, azimuth and radial coordinates of the point at the crystallization front; t_{fr}, time at the crystallization front, sec; L_1, crystallization heat, J/kg; ε, thickness of the solidified skin, m; a_1 and a_2, thermal diffusivities of the molten and hard metals, m^2/sec; T_r^0 and T_t^0, unvaried derivatives of the temperature with respect to the radius and the time; T, temperature function; T_r and T_{rr}, first and second derivatives of the temperature with respect to the radius; T_φ and T_t, first derivatives of the temperature with respect to the angle and the time; $T_{\varphi\varphi}$, second derivative of the temperature with respect to the angle; L, functional or Lagrangian; A, A_1, E, B, A_2, and B_2, constants of integration over the radius; δL, variation of the functional; $f(\varphi)$, function of the angle; $f'(\varphi)$ and $f''(\varphi)$, first and second derivatives with respect to φ; C_1 and C_2, constants of integration over the angle; K, M, P, Q, G_1, G_2, and F_2, constants of integration over r and φ. Subscripts: sur, surface; cr, crystallization; in, initial; fr, front.

REFERENCES

1. F. V. Nedopekin, *Mathematical Modeling of Hydrodynamics and Heat and Mass Transfer in Ingots* [in Russian], Izhevsk (1995).
2. V. V. Sobolev and P. M. Trefilov, *Thermophysics of Solidification of Metal in Continuous Casting* [in Russian], Moscow (1988).
3. G. F. Balandin, *Principles of the Theory of Molding* [in Russian], Pts. 1 and 2, Moscow (1979).
4. B. Ya. Lyubov, in: *Theory of Pool Crystallization* [in Russian], Moscow (1975), pp. 20, 137.
5. V. I. Timoshpol'skii, V. N. Papkovich, I. N. Meleshko, et al., *Izv. Vyssh. Uchebn. Zaved.*, *Énergetika*, Nos. 5–6, 69–73 (1992).
6. R. S. Schechter, *The Variational Method in Engineering* [Russian translation], Moscow (1971).
7. A. V. Luikov, *Heat and Mass Transfer: Handbook* [in Russian], Moscow (1972).
8. E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen* [Russian translation], Moscow (1971).
9. A. E. Kazachkov, *Problems of the Theory and Practice of Steelmaking Production* [in Russian], Moscow (1991).